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QUANTUM MECHANICAL SYSTEM SYMMETRY*

T. J. Tarn,[+] M. Hazewinkel[++] and C. K. Ong[+++]

[+] Department of Systems Science and Mathematics, Box 1040, Washington University, St. Louis, Missouri 63130, USA.

[++] Stichting Mathematisch Centrum, Kruislaan 413, 1098 S J Amsterdam, The Netherlands.

[+++] M/A-COM Development Corporation, M/A-COM Research Center, 1350 Piccard Drive, Suite 310, Rockville, Maryland 20850, USA.

Abstract

The connection of quantum nondemolition observables with the symmetry operators of the Schrödinger equation, is shown. The connection facilitates the construction of quantum nondemolition observables and thus of quantum nondemolition filters for a given system. An interpretation of this connection is given, and it has been found that the Hamiltonian description under which minimal wave pockets remain minimal is a special case of our investigation.

1. Introduction

In developing the theory of quantum nondemolition observables, it has been assumed that the output observable is given. The question of whether or not the given output observable is a quantum nondemolition filter has been answered in [1,2]. In practice, however, only the dynamical equation governing the behavior of the state is known at the outset, and the question is one of the existence and determination of quantum nondemolition observables for the system. This question will now be addressed by appealing to the theory of symmetry groups for the solution of partial differential equations through separation of variables.

The dynamical equation may be written symbolically as

$$\mathcal{S}\psi = 0 ,$$

where $\mathcal{S}$ is the Schrödinger operator. In a more general context, $\mathcal{S}$ may be a linear or nonlinear differential or integral operator. Within this context, group theoretic methods have been used to describe in a systematic manner the possible coordinate systems in which the equation admits solutions via separation of variables. The connection between such methods and quantum nondemolition observables will now be explored. It will be shown that the symmetry operators of the Schrödinger equation are in a sense quantum nondemolition (QND) operators. Conversely, symmetry operators permit an appealing interpretation of QND operators in terms of coordinate transformations.

2. Definitions and Terminologies

The quantum system of interest is described by the Schrödinger equation

$$i\hbar \frac{\partial\psi(x,t)}{\partial t} = H(x,t) , \tag{1}$$

where ψ is the wave function of the system and x an appropriate set of dynamical coordinates.

Definition 1. The symmetry algebra of the Hamiltonian H of a quantum system is generated by those operators that commute with H and possess together with H a common dense invariant domain $\mathcal{D}$.

With the above definition, if $[H,X_1] = 0$ and $[H,X_2] = 0$, then $[H,[X_1,X_2]] = 0$ also on $\mathcal{D}$. Clearly all constants of the motion belong to the symmetry algebra of the Hamiltonian.

Denote the Schrödinger operator by

$$\mathcal{S} \overset{\Delta}{=} i\hbar \frac{\partial}{\partial t} - H .$$

Then the Schrödinger equation (1) can be written as

$$i\hbar \frac{\partial\psi}{\partial t} - H\psi = \mathcal{S}\psi = 0 . \tag{2}$$

Definition 2. If there exist operators S_i, i = 1,...,r, forming a Lie algebra $\mathcal{G}$ such that on the space of solutions of (2)

$$[\mathcal{S},S_i] = f(\mathcal{S}) , \ i = 1,...,r ,$$

where f is a polynomial with analytic coefficients depending on the coordinates and such that f(0) = 0, then the Lie algebra $\mathcal{G}$ is called the dynamical Lie algebra.

Note that if f is linear, then $S \in \mathcal{G}$ is a symmetry operator of $\mathcal{S}$. If ψ is a solution of (2), so is $S\psi$. In general, the dynamical Lie algebra of the quantum system contains time-dependent operators S(t), which, on the space of solutions, satisfy the Heisenberg equation

$$\left[i\hbar\frac{\partial}{\partial t}, S(t)\right] = [H,S(t)] .$$

The subalgebra $\mathcal{G}'$ of time-independent operators satisfies [H,S] = 0 and is the symmetry algebra of H considered in Definition 1.

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3. Sufficient Condition for a Quantum Nondemolition Observable

The observables

$$X_1 = \hat{x} \cos \omega t - (\hat{p}/m\omega) \sin \omega t \ ,$$

$$X_2 = \hat{x} \sin \omega t + (\hat{p}/m\omega) \cos \omega t \ ,$$

where $\hat{p}$ is the momentum operator and $\hat{x}$ is the position operator; of the simple harmonic oscillator introduced in [3,4] are quantum nondemolition observables because of the special form that they take. This special form is a consequence of the fact that X_1 and X_2 are conserved, i.e., in the Heisenberg picture,

$$\frac{dX_{1,2}}{dt} = \frac{\partial X_{1,2}}{\partial t} - \frac{i}{\hbar} [X_{1,2}, H] = 0 \ .$$

It follows immediately that $X_{1,2}$ takes the form $X_{1,2}(t) = f(X_{1,2}(t_o); t, t_o)$. The question that arises naturally is whether or not X_1 and X_2 are the only continuous quantum nondemolition observables for the simple harmonic oscillator, and, if not, how one can determine the other quantum nondemolition observables, and in particular examples that are not conserved.

To determine the quantum nondemolition observables for an arbitrary quantum system, we first seek a sufficient condition under which an observable C qualifies as a quantum nondemolition observable. To distinguish between the Heisenberg and Schrödinger pictures, subscripts H and S will be used.

Proposition 1. A sufficient condition for a self-adjoint operator C to be a QND operator is that

$$[\mathcal{S}, C] = f(\mathcal{S}) + g(C) \ ,$$

where f and g are polynomials with analytic coefficients depending on the coordinates and such that $f(0) = g(0) = 0$.

Proof. Let $\psi_S(0)$ be an eigenstate of $C_S(0)$ with corresponding eigenvalue $\lambda(0)$, i.e.,

$$C_S(0)\psi_S(0) = \lambda(0)\psi_S(0) = \lambda_o \psi_S(0) \ .$$

Let $\psi_S(t)$ be the solution of (2) with initial condition $\psi_S(0)$.

Case 1. Consider $g \equiv 0$. Then $C_S\psi_S$ is also a solution for (2). Since $C_S(t)\psi_S(t)$ and $\lambda(0)\psi_S(t)$ are, by Cooper's theorem [5,6], both solutions with the same initial condition, we conclude that

$$C_S(t)\psi_S(t) = \lambda(0)\psi_S(t) \ .$$

Thus $\psi_S(t)$ remains an eigenstate of $C_S(t)$ with the same eigenvalue $\lambda(0)$. By Remark 1 of Section 3.1 in [2], C(t) is a QND operator of the Hamiltonian H.

Case 2. If $g(C_S) \neq 0$, then

$$\mathcal{S} C_S \phi_S = g(C_S)\phi_S \ , \tag{3}$$

where ϕ_S is an arbitrary element of the solution space of (2). We note that (3) can be written as

$$i\hbar \frac{\partial \hat{\psi}_S}{\partial t} = (H + g'(C_S))\hat{\psi}_S \ , \tag{4}$$

with $\hat{\psi}_S = C_S \phi_S$, and $g'(0)$ not necessarily zero.

Now a simple computation shows that

$$\hat{\psi}_S = e^{-i/\hbar g'(\lambda_o)t} \psi_S$$

is a solution of (4) with the proper initial condition corresponding to the initial C_S-measurement. So, apart from a phase shift, the conclusion remains the same as in Case 1, and C_S is a QND operator for the given Hamiltonian H as well as $H + g'(C_S)$.

Remark 1. More generally, the result holds for g(C) replaced by $g(C, B_1, \ldots, B_n)$, where $C, B_1, \ldots, B_n$ commute with one another.

Remark 2. If $g \equiv 0$, we have a conserved QND operator. This is seen as follows. By a unitary transformation, $[\mathcal{S}, C_S] = f(\mathcal{S})$ implies in the Heisenberg picture that

$$\frac{dC_H}{dt} = i\hbar \left(\frac{\partial C_H}{\partial t} + \frac{i}{\hbar} [H_H, C_H] \right)\psi = 0 \ .$$

For time-independent observables, the latter relation reduces to

$$[H_H, C_H] = 0 \ ,$$

i.e., constants of the motion are QND operators.

4. Computation of QND Operators

From Proposition 1 of the preceding section, the symmetry operators are seen to be QND operators (operators rather than observables since the symmetry operators need not be self-adjoint). To compute QND operators for a given system, we can follow the standard procedure for computing symmetry operators [7,8]. However, for QND operators the following modifications are to be made:

(1) The operator $i\hbar(\partial/\partial t)$ is to be replaced by the Hamiltonian H in the final result.

(2) The resulting operators are to be extended to self-adjoint operators whenever possible, i.e., when the deficiency indices [9] are equal.

In the following examples, we set $\hbar = 1$ and also put $m = 1$.

Example 1. (Free particle)

The wavefunction $\psi(x,t)$ of a free particle is a solution of

$$\frac{1}{i} \frac{\partial}{\partial t}\psi(x,t) = \frac{1}{2} \frac{\partial^2}{\partial x^2}\psi(x,t) \ .$$

Here $S \equiv (1/i)(\partial/\partial t) - (1/2)(\partial^2/\partial x^2)$. Computing the first-order symmetry operators of S, we obtain (see also [10])

$$S_1 = I , \quad S_2 = -\frac{\partial}{\partial x} , \quad S_3 = -t\frac{\partial}{\partial x} + ix ,$$

$$S_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \frac{1}{2} ,$$

$$S_5 = -(t^2+1)\frac{\partial}{\partial t} - tx\frac{\partial}{\partial x} + \frac{1}{2}(ix^2 - t - i) ,$$

$$S_6 = -(t^2-1)\frac{\partial}{\partial t} - tx\frac{\partial}{\partial x} + \frac{1}{2}(ix^2 - t) .$$

From the correspondence rules

$$x \to \hat{x} , \ -i(\partial/\partial x) \to \hat{p} ,$$

and

$$i(\partial/\partial t) \to H = 1/2\,\hat{p}^2,$$

the associated QND operators are determined as

$$C_1 = I \text{ (trivial)} , \quad C_2 = \hat{p} , \quad C_3 = t\hat{p} - \hat{x} ,$$

$$C_4 = t\hat{p}^2 - \hat{x}\hat{p} \qquad \text{(nonself-adjoint)} ,$$

$$C_5 = (t^2+1)\hat{p}^2 - t\hat{x}\hat{p} + \frac{1}{2}\hat{x}2 \qquad \text{(nonself-adjoint)} ,$$

$$C_6 = (t^2-1)\hat{p}^2 - t\hat{x}\hat{p} + \frac{1}{2}\hat{x}^2 \qquad \text{(nonself-adjoint)} .$$

Example 2. (Particle in a Constant External Field)

For a particle subject to a constant external field F, the Schrödinger equation is given by

$$\frac{1}{i}\frac{\partial}{\partial t}\psi = \frac{1}{2}\frac{\partial^2}{\partial x^2}\psi + Fx\psi$$

with

$$S = \frac{1}{i}\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2} - Fx .$$

The symmetry operators are found to be

$$S_1 = I , \quad S_2 = -\frac{\partial}{\partial x} + \frac{1}{2} iFt ,$$

$$S_3 = -t\frac{\partial}{\partial x} + ix + \frac{1}{2} iFt^2 ,$$

$$S_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \frac{1}{2} - \frac{1}{2} ixFt ,$$

$$S_5 = -(t^2+1)\frac{\partial}{\partial t} - tx\frac{\partial}{\partial x} + \frac{1}{2}(ix^2-t-i) + \frac{1}{2} ixFt^2,$$

$$S_6 = -(t^2-1)\frac{\partial}{\partial t} - tx\frac{\partial}{\partial x} + \frac{1}{2}(ix^2-t) + \frac{1}{2} ixFt^2.$$

The corresponding QND operators are

$$C_1 = I \text{ (trivial)} , \quad C_2 = \hat{p} , \quad C_3 = t\hat{p} - \hat{x} ,$$

$$C_4 = t\hat{p}^2 - \hat{x}\hat{p} - \frac{3}{2}\hat{x}Ft \qquad \text{(nonself-adjoint)} ,$$

$$C_5 = \left(\frac{t^2+1}{2}\right)\hat{p}^2 - t\hat{x}\hat{p} + \frac{1}{2}\hat{x}^2 - \frac{1}{2}\hat{x}F \text{ (nonself-adjoint)},$$

$$C_6 = \left(\frac{t^2-1}{2}\right) p^2 - t\hat{x}\hat{p} + \frac{1}{2}\hat{x}^2 + \frac{1}{2}\hat{x}F \text{ (nonself-adjoint)} ,$$

Example 3. (Simple Harmonic Oscillator)

The wave function $\psi(x,t)$, $x \in R$, $t > 0$, of the Simple Harmonic Oscillator is a solution of the Schrödinger equation

$$\frac{1}{i}\frac{\partial}{\partial t}\psi = \frac{1}{2}\frac{\partial^2}{\partial x^2}\psi - \frac{1}{2}\omega^2 x^2\psi .$$

The symmetry operators for

$$S \equiv \frac{1}{i}\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\omega^2 x^2$$

have been computed in [11] with the results

$$S_1 = I ,$$

$$S_2 = -\cos\omega t\frac{\partial}{\partial x} - i\omega x\sin\omega t ,$$

$$S_3 = -\frac{\sin\omega t}{\omega}\frac{\partial}{\partial x} + ix\cos\omega t ,$$

$$S_4 = x\cos 2\omega t\frac{\partial}{\partial x} + \frac{\sin 2\omega t}{\omega^2}\frac{\partial}{\partial t} + i\omega x^2\sin\omega t + \cos^2\omega t - \frac{1}{2} ,$$

$$S_5 = \left[\frac{x\sin 2\omega t}{\omega}\frac{\partial}{\partial x} + \frac{2\sin^2\omega t}{\omega^2}\frac{\partial}{\partial t} - ix^2\cos 2\omega t + \frac{1}{2\omega}\sin 2\omega t\right]\frac{(1-\omega^2)}{2} + \frac{\partial}{\partial t} + \frac{1}{2} i ,$$

$$S_6 = \left[\frac{x\sin 2\omega t}{\omega}\frac{\partial}{\partial x} + \frac{2\sin^2\omega t}{\omega^2}\frac{\partial}{\partial t} - ix^2\cos 2\omega t + \frac{1}{2\omega}\sin 2\omega t\right]\frac{(1+\omega^2)}{2} - \frac{\partial}{\partial t} .$$

Invoking the correspondence rules

$$x \to \hat{x} , \ -i(\partial/\partial x) \to \hat{p} , \ i(\partial/\partial t) \to \frac{1}{2}\hat{p}^2 + \omega^2\hat{x}^2,$$

the associated QND operators are

$$C_1 = I \text{ (trivial)} ,$$

$$C_2 = \hat{x}\sin\omega t + \frac{\cos\omega t}{\omega}\hat{p} ,$$

$$C_3 = \hat{x}\cos\omega t - \frac{\sin\omega t}{\omega}\hat{p} ,$$

$$C_4 = -\cos 2\omega t\, \hat{x}\hat{p} + \frac{\sin 2\omega t}{\omega}\left(\frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2 x^2\right) - \hat{x}^2 \sin 2\omega t \qquad \text{(nonself-adjoint)} ,$$

$$C_5 = \frac{(1-\omega^2)}{2}\left[-\frac{\sin 2\omega t}{\omega}\hat{x}\hat{p} + \frac{2\sin^2\omega t}{\omega^2}\left(\frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2\right) + \hat{x}^2\cos\omega t\right] + \left(\frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2 x^2\right) \text{(nonself-adjoint)} ,$$

$$C_6 = \frac{(1+\omega^2)}{2}\left[-\frac{\sin 2\omega t}{\omega}\hat{x}\hat{p} + \frac{2\sin^2\omega t}{\omega^2}\left(\frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2\right) + \hat{x}^2\cos 2\omega t\right] - \left(\frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2\right) \qquad \text{(nonself-adjoint)} .$$

Note that C_3 and C_2 are respectively the QND observables X_1 and X_2 introduced in Section 3.

In all of the above examples $S_1, S_2, \ldots, S_6$ form a basis for the dynamical Lie algebra. They are skew-symmetric and the necessary steps have been taken to make $C_1, C_2, \ldots, C_6$ symmetric. Depending on how the external force is coupled to the system, some of the QND operators so derived may turn out to be quantum nondemolition filters as well. It can be shown [7] that the symmetry algebras obtained in Examples 1 to 3 are isomorphic.

5. An Interpretation

The key idea in quantum nondemolition measurements is the notion of phase sensitivity [12]. In a phase-sensitive measurement, the fluctuations are not allowed to be randomly distributed in phase. This is easily seen using the "squeeze" operator introduced for the simple harmonic oscillator by Stoler [13,14] and Hollenhorst [15]:

$$\tilde{S}(z) \equiv \exp\left[\frac{1}{2} z(a^\dagger)^2 - \frac{1}{2} z^*(a)^2\right].$$

The squeeze operator $\tilde{S}(z)$ is unitary, and if $|\psi\rangle$ is a state of the system and z a real number r, then

$$\tilde{S}^\dagger(r)\hat{x}\,\tilde{S}(r) = e^r\,\hat{x} ,$$

so that

$$\langle\psi|\tilde{S}^\dagger(\hat{x})^k\tilde{S}\psi\rangle = \langle\psi|e^{rk}\,\hat{x}^k\,\psi\rangle = e^{rk}\,\langle\psi|x^k\rangle .$$

Similarly,

$$\tilde{S}^\dagger(r)\hat{p}\,\tilde{S}(r) = e^{-r}\,\hat{p}$$

and

$$\langle\psi|\tilde{S}^\dagger(\hat{p})^k\tilde{S}|\psi\rangle = \langle\psi|e^{-rk}\,\hat{p}^k\psi\rangle = e^{-rk}\langle\psi|\hat{p}^k\psi\rangle .$$

Therefore $\tilde{S}(r)|\psi\rangle$, for large $r > 0$, represents a state highly localized in momentum space or, for large $r < 0$, highly localized in position. The reason for the name "squeeze" operator is now apparent.

The squeezed state $\tilde{S}(z)|0\rangle$, where $|0\rangle$ denotes the ground state, can be generalized to wave packets with the same shape but displaced from the origin in the position and momentum space by

$$|\beta,z\rangle = D(\beta)\tilde{S}(z)|0\rangle ,$$

where $D(\beta) = \text{esp}(\beta\hat{a}^\dagger - \beta^*\hat{a})$ is the displacement operator [16]. These states develop in time according to

$$e^{-i\omega\hat{a}^\dagger\hat{a}}|\beta,z\rangle = |\beta e^{-i\omega t}, ze^{-2i\omega t}\rangle .$$

The dispersions of $\hat{x}$ and $\hat{p}$ for the simple harmonic oscillator in this state are given by [15].

$$\Delta\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\left\{\alpha^2\sin^2\omega t + \left(\frac{1}{\alpha^2}\right)\cos^2\omega t\right\}^{1/2} .$$

$$\Delta\hat{p} = \left(\frac{m\hbar\omega}{2}\right)^{1/2}\left\{\left(\frac{1}{\alpha^2}\right)\sin^2\omega t + \alpha^2\cos^2\omega t\right\}^{1/2} ,$$

where $\alpha = e^{-r}$. We see from the above expressions that at time $t = 0$, $\hat{x}$ can be measured arbitrarily precisely as $\alpha \to \infty$, while at $t = \pi/2\omega$, $\hat{p}$ can be measured arbitrarily precisely at $\alpha \to \infty$. Measurement of the time-varying operator X_1 allows a precise measurement of a linear combination of $\hat{x}$ and $\hat{p}$ to be made by suitably tracking the squeezed state. Note that at $t = 0$ a measurement of X_1 corresponds to a position measurement, while at $t = \pi/2\omega$ it corresponds to a momentum measurement; and these measurements are dispersion-free as $\alpha \to \infty$. In fact the dispersion of X_1 is given simply by

$$\Delta X_1 = \frac{1}{\alpha} .$$

A similar argument can be carried out for X_2, the corresponding dispersion being

$$\Delta X_2 = \alpha .$$

A pictorial representation of the above description is given in [4].

A given separable coordinate system for a partial differential equation corresponds to a symmetry operator. The separated solution is characterized as an eigenfunction of a symmetry operator, the eigenvalue playing the role of the separation constant. In the new coordinate system {u,v}, the symmetry operator transforms to $\partial/\partial u$ and the separated solution takes the form U(u)V(v). (More generally the solution is R-separable and takes the form exp(iR(u,v))U(u)V(v) [7].) The fact that the solution separates indicates that random fluctuations are squeezed out, thus making the corresponding state more susceptible to a phase-sensitive measurement. Indeed, if one were to measure the symmetry operator (assuming it is an observable) in such a state, then the measurement result would be the separation constant. By construction the symmetry operators given in Examples 1-3 are skew-symmetric. From spectral theory

[17], we know that to each skew-symmetric operator S, there corresponds a one-parameter unitary group $U(\alpha) = \exp(\alpha S)$. It turns out that elements of the unitary group associated with the dynamical Lie algebra at t = 0 can be interpreted as squeeze operators.

Example 1. Consider the symmetry algebra for the simple harmonic oscillator given in Example 3 of Section 4. We now think of symmetry operators at a fixed time, say t = 0. Taking $\omega = 1$, the symmetry operators become

$$S_1 = I \ , \qquad S_2 = -\frac{\partial}{\partial x} \ ,$$

$$S_3 = ix \ , \qquad S_4 = x\frac{\partial}{\partial x} + \frac{1}{2} \ ,$$

$$S_5 = ix^2 - \frac{1}{2}\frac{\partial^2}{\partial x^2} \ , \qquad S_6 = \frac{1}{2} i \frac{\partial^2}{\partial x^2} \ ,$$

where S_5 and S_6 are obtained by substituting

$$i\frac{\partial}{\partial t} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - x^2\right).$$

The above operators exponentiate to unitary operators. The unitary group corresponding to S_1 is the gauge transformation, while that corresponding to S_2 is the shift or translation. The one-parameter unitary group corresponding to S_3 is conjugate to the shift via the Fourier transform. The operator S_4 gives rise to the group of dilations (or tensions) τ_α: $f(x) \to (\tau_\alpha f)(x) = e^{\alpha/2} f(e^\alpha x)$. We have already seen this in the guise of the squeeze operator defined by Hollenhorst. Corresponding to S_5 is the group of Fourier-Mehler Transforms. More details concerning the above one-parameter groups can be found in [11]. The unitary operator $\exp(i\alpha S_6)$ is the operator of time translation [18].

With reference to Section 4, we note that the symmetry algebras in Examples 1-3 are isomorphic to one another, and therefore the above comments apply to Examples 1 and 2 as well.

Squeezed states are closely related to coherent states. Stoler [13] has shown that they are unitarily equivalent via the squeeze operator. Stoler calls squeezed states "minimum uncertainty wave packets" or "minimal packets." In general, a minimal packet at t = 0 does not remain one with the passage of time. Mehta, Chand, Sudarshan and Vedam [19,20] have derived the most general form of the Hamiltonian such that coherent states remain coherent states at all times. Similarly, Stoler has determined the most general Hamiltonian that preserves general minimal packets. Since the minimal packets are eigenvectors of some operators, albeit nonself-adjoint, we see that the notion of QND operators allows an amplification on these considerations. For example, the persistent coherent states are eigenvectors of the annihilation operator $\hat{a}$, and therefore one can interpret $\hat{a}$ as a QND operator for the Hamiltonian determined by Mehta et al. Likewise the minimal packets of Stoler are eigenvectors of the operator $\tilde{S}(z)\hat{a}\,\tilde{S}^\dagger(z)$, where $\tilde{S}(z)$ is the squeeze operator.

Example 2. Let the Hamiltonian be

$$H(t) = \omega(t)\hat{a}^\dagger\hat{a} + f(t)\hat{a}^\dagger + f^*(t)\hat{a} + \beta(t) \ .$$

Using criteria given in [2] one can easily verify that $\hat{a}$ is a QND operator. Consequently an eigenvector of $\hat{a}$ remains one during subsequent evolution. Indeed H(t) is the most general form of the Hamiltonian for a simple harmonic oscillator in the presence of interaction under the restriction that states that are initially coherent remain coherent at all times [19].

6. Conclusion

The connection between symmetry operators of the Schrödinger equation and QND operators has been demonstrated in this chapter. Drawing on the mathematical results on the symmetry operators, one can construct QND operators for a given system. On the other hand, the study of QND operators provides a physical interpretation for the solution of partial differential equations by separation of variables. The unitary groups that arise out of the symmetry operators can be interpreted in terms of squeeze operators. In particular, the element of the dilation group are just the squeeze operator introduced by Stoler [13,14], Hollenhoerst [15], Yuen [21]. We also saw that if we allow nonself-adjoint QND operators, then the dynamics of coherent states or minimal packets are encompassed in the present theory.

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